Segments of Circles: Theorems for secants and tangents

Lesson Plan by Kara Painter

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| Goals & Objectives |
| 1. Students will discover and prove: If two tangent segments are drawn to a circle from an external point, then both tangent segments are congruent. 2. Students will discover and prove: If a tangent segment and a secant segment are drawn to a circle from an exterior point, then the square of the measure of the tangent segment is equal to the product of the measures of the secant segment and its external secant segment. 3. Students will discover and prove: If two secant segments are drawn to a circle from an exterior point, then the product of the measures of one secant segment and its external secant segment is equal to the product of the measures of the other secant segment and its external secant segment. 4. Students will be able to solve algebraic examples of the above theorems. |
| Illinois State Learning Standards - Mathematics |
| **9.A.5**  Use geometric figures and their properties to solve problems in the arts, the physical and life sciences and the building trades, with and without the use of technology.  **9.B.4**  Recognize and apply relationships within and among geometric figures.  **9.C.4a**  Construct and test logical arguments for geometric situations using technology where appropriate.  **9.C.4b**  Construct and communicate convinc­ing arguments for geometric situations.  **9.C.4c**  Develop and communicate mathemati­cal proofs (e.g., two-column, paragraph, indirect) and counter examples for geometric statements.  **9.D.4**  Analyze and solve problems involving triangles (e.g., distances which cannot be measured directly) using trigonometric ratios. |
| Sequence of Activities |
| 1. Student discovery of theorems on Geometer’s Sketchpad. (Worksheet #1)    1. Have students log onto a computer and open a new Geometer’s Sketchpad document    2. Have students individually go through Worksheet #1 and discover the theorems.    3. Once all students have finished the Geometer’s Sketchpad discovery, come together as a class and make sure all students have discovered the correct theorems 2. As a class, prove the following three theorems using similar and congruent triangles    * 1. If two tangent segments are drawn to a circle from an external point, then both tangent segments are congruent.      2. If a tangent segment and a secant segment are drawn to a circle from an exterior point, then the square of the measure of the tangent segment is equal to the product of the measures of the secant segment and its external secant segment.      3. If two secant segments are drawn to a circle from an exterior point, then the product of the measures of one secant segment and its external secant segment is equal to the product of the measures of the other secant segment and its external secant segment. 3. Go through three examples of algebraic problems involving the above theorems 4. Distribute Homework Worksheet to every student. |
| Materials Needed |
| 1. Copies of Worksheet #1 for each student 2. Copies of Worksheet #2 for each student 3. Copies of Worksheet #3 for each student 4. Copies of Homework Worksheet for each student 5. A classroom set of computers with Geometer’s Sketchpad installed. |

Discovery with Geometer’s Sketchpad – Student Worksheet

Worksheet #1

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| Tangent – Tangent Discovery |
| 1. Create a circle of any size toward the middle of the screen, call it circle O 2. Create a point external to the circle, call it point A 3. Create a line through point A and tangent to circle O. Call the point of tangency point B. 4. Create another line through point A and tangent to circle O (it can not be the same line created in step 3). Call the point of tangency point C. 5. Using the segment tool, create a segment from points A and B, and a segment from points A and C. 6. Calculate the measure of segment AB and AC. 7. Move point A and observe what happens to the lengths of AB and AC. |
| 1. What do you discover? |
| Tangent – Secant Discovery |
| 1. Go to “File” then “Document Options” and rename page 1 as “TT” 2. Add a new page and rename it “TS” 3. You should now see the two different pages on the bottom by the scrollbar 4. Click on TS page. 5. Create a circle of any size toward the middle of the screen, call it circle O 6. Create a point external to the circle, call it point A 7. Create a line through point A and tangent to circle O. Call the point of tangency point B. 8. Create any secant through point A. Call the points of intersection C and D, with C between points A and D. 9. Using the segment tool, create a segment from points A and B, a segment from points A and C and a segment from points A and D. 10. Calculate the measure of segment AB, AC and AD. 11. Calculate AB2 and AC • AD. 12. Move point A and observe what happens to the lengths of AB2 and AC • AD |
| 1. What do you discover? |
| Secant – Secant Discovery |
| 1. Go to “File” then “Document Options” and add a new page. Call it “SS” 2. You should now see the three different pages on the bottom by the scrollbar 3. Click on SS page. 4. Create a circle of any size toward the middle of the screen, call it circle O 5. Create a point external to the circle, call it point A 6. Create any secant through point A. Call the points of intersection B and C, with B between points A and C. 7. Create any other secant through point A. (it can not be the same secant created in step 6). Call the points of intersection D and E, with D between points A and E. 8. Using the segment tool, create a segment from points A and B, a segment from points A and C, a segment from points A and D and a segment from points A and E. 9. Calculate the measure of segment AB, AC, AD and AE. 10. Calculate AB • AC and AD • AE. 11. Move point A and observe what happens to the lengths of AB2 and AC • AD |
| 1. What do you discover? |

Proof of Circle Theorems – Student Worksheet

Worksheet #2

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| Tangent – Tangent Proof  If two tangent segments are drawn to a circle from an external point, then both tangent segments are congruent. |
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| Secant – Secant Proof  If two secant segments are drawn to a circle from an exterior point, then the product of the measures of one secant segment and its external secant segment is equal to the product of the measures of the other secant segment and its external secant segment. |
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| Tangent – Secant Proof  If a tangent segment and a secant segment are drawn to a circle from an exterior point, then the square of the measure of the tangent segment is equal to the product of the measures of the secant segment and its external secant segment. |
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Proof of Circle Theorems – Teacher copy

Worksheet #2

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| Tangent – Tangent Proof  If two tangent segments are drawn to a circle from an external point, then both tangent segments are congruent. |
| Given Tangents AB and AC, we need to construct segments BD, AD and DC. By definition of tangent lines, the radius will always be perpendicular to the tangent. Hence ∠ABC and ∠ACD are right angles and congruent to each other. BD and DC are radii so they are congruent and AD is a reflexive segment for ΔABD and ΔACD. So we can say ΔBDA ≅ ΔCDA by Right angle – Hypotenuse – Leg Theorem. So then AB ≅ AC, by Corresponding Parts of Congruent Triangles are Congruent (CPCTC). |

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| Secant – Secant Proof  If two secant segments are drawn to a circle from an exterior point, then the product of the measures of one secant segment and its external secant segment is equal to the product of the measures of the other secant segment and its external secant segment. |
| ∠ABC is an inscribed angle with intercepted arc ADC. Therefore, m∠ABC = ½(measure of arc ADC). ∠ADC is also an inscribed angle but has intercepted arc ABC. Therefore, m∠ADC = ½(measure of arc ABC). Since m(arc ABC) + m(arc ADC) = 360, thus making m∠ABC + m∠ADC = 180. Thus, ∠ABC and ∠ADC are supplementary. By the picture, we see that ∠PBC and ∠ABC form a linear pair and are therefore also supplementary. So if ∠ABC is supplementary to ∠ADC and ∠PBC is supplementary to ∠ABC, then ∠PBC ≅ ∠ADC. This is because angles supplementary to the same angle are congruent. We also know that ∠BPC is reflexive for ΔBPC and ΔDPA. So by Angle Angle similarity, ΔBPC is similar to ΔDPA. By definition of similar triangles, the corresponding sides must be proportional. So the following proportion of the corresponding sides is true: . By cross multiplying, we see (PB)(PA) = (PC)(PD). |

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| Tangent – Secant Proof  If a tangent segment and a secant segment are drawn to a circle from an exterior point, then the square of the measure of the tangent segment is equal to the product of the measures of the secant segment and its external secant segment. |
| This theorem is basically just an extension of the Secant – Secant Theorem above. We needed to construct TW and TX. ∠TWX is the inscribed angle to its intercepted arc TX, so therefore, m∠TWX = (1/2)(measure of arc TX). We also have learned earlier that ∠PTX is (1/2)(measure of arc TX). (See below you or the students forget the proof) We also know ∠TPX is a reflexive angle. Therefore, triangle TPX is similar to triangle WPT. So then we can conclude the following proportion: . If we cross multiply, we get (PT)2 = (PX)(PW).  If students forget: Proof of why ∠PTX is (1/2)(measure of arc TX)  Simplify the diagram so students can focus on the given information. Given the diagram to the left, we have tangent AD and secant AB. So if we construct segment AC and CB, they are both radii and hence congruent. So triangle ACB is isosceles and hence the base angles ∠BAC and ∠ABC are congruent. Let’s call their measurement x for simplicity. Therefore, we can say m∠ACB = 180 – 2x = 2(90 – x). We also know ∠CAD is a right angle because the radii and tangent are always perpendicular. So we can say m∠BAD = 90 – x. So m∠BAD is half m∠ACB. |

Circle Theorems – In Class Practice

Worksheet #3

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| Problems | Solutions |
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Circle Theorems – In Class Practice

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| Problems | Solutions |
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Circle Theorems Homework Worksheet

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| 3. | 4. |
| 5. | 6. |
| 7. | 8. |
| 9. | 10. |
| 13. | 14. |

**Self Reflection**

This lesson is one of many that should be incorporated into a discovery of circles chapter. Students should have a working knowledge of basic circle properties, including vocabulary terms such as, secant, tangent, and point of tangency. Students should also understand and be able to prove triangles similar. This lesson focuses heavily on discovering the three theorems. Some teachers may prefer to spread this lesson out across two days and spend more time on problems that apply these theorems. The proofs of these theorems combine circle properties and similar triangles, concepts that should have already been covered in class, however, the proofs may be too much for some Geometry courses. Individual teachers would have to see how these proofs would fit best in their curriculum. One topic this lesson plan does not include is the theorem for finding length of segments of chords. That topic could be used to extend this lesson in class or even given to students as a self-discovery homework problem. A natural progression of this topic would lead into the theorems dealing with the angles created by the secants, tangents and chords of a circle.

**References**

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**STATE GOAL 9: Use geometric methods to analyze, categorize and draw conclusions about points, lines, planes and space.**

**Why This Goal Is Important:** Geometry provides important methods for reasoning and solving problems with points, lines, planes and space. The word “geometry” comes from Greek words meaning “measurement of the Earth.” While we use modern technology and employ a wider variety of mathematical tools today, we still study geometry to understand the shapes and dimensions of our world. The applications of geometry are widespread in construction, engineering, architecture, mapmaking and art. Historically, geometry is a way to develop skill in forming convincing arguments and proofs. This goal of developing a means of argument and validation remains an important part of our reasons for studying geometry today.

A. Demonstrate and apply geo­metric concepts involving points, lines, planes and space.

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| **Early Elementary** | **Late Elementary** | **Middle/Junior High School** | **Early High School** | **Late High School** |
| **9.A.1a** Identify related two- and three-dimensional shapes including circle-sphere, square-cube, triangle-pyramid, rectangle-rectangular prism and their basic properties. | **9.A.2a** Build physical models of two- and three-dimensional shapes. | **9.A.3a** Draw or construct two- and three- dimensional geometric figures including prisms, pyramids, cylinders and cones. | **9.A.4a** Construct a model of a three-dimensional figure from a two-dimensional pattern. | **9.A.5** Use geometric figures and their properties to solve problems in the arts, the physical and life sciences and the building trades, with and without the use of technology. |
| **9.A.1b** Draw two-dimensional shapes. | **9.A.2b** Identify and describe how geometric figures are used in practical settings (e.g., construction, art, advertising). | **9.A.3b** Draw transformation images of figures, with and without the use of technology. | **9.A.4b** Make perspective drawings, tessella­tions and scale drawings, with and without the use of technology. |  |
|  | **9.A.2c** Describe and draw repre­sen­ta­tions of geometric relationships, patterns, symmetries, and designs in two- and three-dimensions with and without technology. | **9.A.3c** Use concepts of symmetry, congru­ency, similarity, scale, perspective, and angles to describe and analyze two- and three-dimensional shapes found in practical applica­tions (e.g., geodesic domes, A-frame houses, basketball courts, inclined planes, art forms, blueprints). |  |  |

B. Identify, describe, classify and compare relationships using points, lines, planes and solids.

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| **Early Elementary** | **Late Elementary** | **Middle/Junior High School** | **Early High School** | **Late High School** |

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| **9.B.1a** Identify and describe characteristics, similarities and differences of geometric shapes. | **9.B.2** Compare geometric figures and determine their properties including parallel, perpendicular, similar, congruent and line symmetry. | **9.B.3** Identify, describe, classify and compare two- and three- dimensional geometric figures and models according to their properties. | **9.B.4** Recognize and apply relationships within and among geometric figures. | **9.B.5** Construct and use two- and three-dimensional models of objects that have practical applications (e.g., blueprints, topo­graphical maps, scale models). |
| **9.B.1b** Sort, classify and compare familiar shapes. |  |  |  |  |
| **9.B.1c** Identify lines of symmetry in simple figures and construct symmetrical figures using various concrete materials. |  |  |  |  |

C. Construct convincing argu­ments and proofs to solve problems.

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| **Early Elementary** | **Late Elementary** | **Middle/Junior High School** | **Early High School** | **Late High School** |
| **9.C.1** Draw logical conclusions and com­muni­cate reasoning about simple geometric figures and patterns using concrete materials, diagrams and contemporary technology. | **9.C.2** Formulate logical arguments about geometric figures and patterns and communicate reasoning. | **9.C.3a** Construct, develop and communicate logical arguments (informal proofs) about geometric figures and patterns. | **9.C.4a** Construct and test logical arguments for geometric situations using technology where appropriate. | **9.C.5a** Perform and describe an original investigation of a geometric problem and verify the analysis and conclusions to an audience. |
|  |  | **9.C.3b** Develop and solve problems using geometric relationships and models, with and without the use of technology. | **9.C.4b** Construct and communicate convinc­ing arguments for geometric situations. | **9.C.5b** Apply physical models, graphs, coordinate systems, networks and vectors to develop solutions in applied contexts (e.g., bus routing, areas of irregular shapes, describing forces and other physical quantities). |
|  |  |  | **9.C.4c** Develop and communicate mathemati­cal proofs (e.g., two-column, paragraph, indirect) and counter examples for geometric statements. |  |

D. Use trigonometric ratios and circular functions to solve problems.

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| **Early Elementary** | **Late Elementary** | **Middle/Junior High School** | **Early High School** | **Late High School** |
| **[BLANK]** | **[BLANK]** | **9.D.3** Compute distances, lengths and measures of angles using proportions, the Pythagorean theorem and its converse. | **9.D.4** Analyze and solve problems involving triangles (e.g., distances which cannot be measured directly) using trigonometric ratios. | **9.D.5** Analyze and solve problems involving periodic patterns (e.g., sound waves, tide variations) using circular functions and com­municate results orally and in writing. |

